Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 24, N. 1, 1-11

On a Theorem of Fatou

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Abstract. We prove a result on the backward dynamics of a rational function nearby a point not contained in the ω-limit set of a recurrent critical point. As a corollary we show that a compact invariant subset of the Julia set, not containing critical or parabolic points, and not intersecting the ω-limit set of *recurrent* critical points, is expanding, thus extending a classical criteria of Fatou. We also prove that the boundary of a Siegel disk is always contained in the ω-limit set of a *recurrent* critical point.

1. The Results

Let $\overline{\mathbf{C}}$ be the Riemann sphere. Given a rational map $f: \overline{\mathbf{C}} \hookleftarrow$, we say that $\Lambda \subset \overline{\mathbf{C}}$ is an *expanding set* of f if it is compact, invariant (i.e. $f(\Lambda) = \Lambda$) and there exists a positive integer $n > 0$ such that $|(f^n)'(z)| > 1$ for every $z \in \Lambda$. A not very difficult to prove, and yet very useful criteria to recognize this fundamental class of invariant sets was given by Fatou (P.Fatou, Sur les équations fonctionelles, Bull. Soc. Math. France 47, 1919, 161–271). He proved that if a compact invariant set Λ of a rational map $f: \overline{\mathbf{C}} \hookleftarrow$, is contained in its Julia set $J(f)$, and the ω-limit set $\omega(c)$ of every critical point of f satisfies $\omega(c) \cap \Lambda = \phi$, then Λ is expanding. Here we shall provide an improvement of this result, that roughly speaking states that one has to be concerned only about intersections with *recurrent* critical points. Recall that a periodic point p of a rational map f is said to be parabolic if $(f^n)'(p)$ is a root of unity when n is the period of p.

Theorem I. *Let $f: \overline{\mathbf{C}} \hookleftarrow$ be a rational map and $\Lambda \subset J(f)$ a compact invariant set not containing critical points or parabolic periodic points.*

Received 10 November 1992.

Then either Λ is expanding or $\Lambda \cap \omega(c) \neq \phi$ for some recurrent critical point c of f.

It follows immediately from Theorem I that every periodic orbit of f, that is not parabolic or a source, is contained in the ω-limit set of a *recurrent* critical point.

This theorem will be a corollary of the following result:

Theorem II. *Let $f: \overline{\mathbf{C}} \hookleftarrow$ be a rational map. If a point $x \in J(f)$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for all $\varepsilon > 0$ there exists a neighborhood U of x such that:*

(a) *For all $n \geq 0$, every connected component of $f^{-n}(U)$ has diameter $\leq \varepsilon$;*

(b) *There exists $N > 0$ such that for all $n \geq 0$ and every connected component V of $f^{-n}(U)$ the degree of f^n/V is $\leq N$;*

(c) *For all $\varepsilon_1 > 0$ there exists $n_0 > 0$ such that every connected component of $f^{-n}(U)$, $n \geq n_0$, has diameter $\leq \varepsilon_1$.*

The core of the theorem is (a), from which properties (b) and (c) will easily follow.

We begin by proving Theorem II. The deduction of Theorem I from Theorem II will be given afterwards, when we shall also prove the following corollary to Theorem II.

Corollary. *If Γ is the boundary of a Siegel disk or a connected component of the boundary of a Herman ring, there exists a recurrent critical point c such that $\omega(c) \supset \Gamma$.*

2. The Proofs

Given an open set $U \subset \overline{\mathbf{C}}$ denote $c(U, n)$ the set of connected components of $f^{-n}(U)$. Observe that $V \in c(U, n)$ implies $f^j(V) \in c(U, n-j)$ for all $0 \leq j \leq n$. If $V \in c(U, n)$ define $\Delta(V, n) = \#\{x \in V \mid (f^n)'(x) = 0\}$ counted with algebraic multiplicity. Given x as is the statement of Theorem II we can assume without loss of generality that $f(\infty) = \infty$ and $\infty \neq x$. Hence, if U is a neighborhood of x not containing ∞, it follows that $\infty \notin f^{-n}(U)$ for all $n \geq 0$. This means that from now on we shall have to deal only with subsets of the complex plane $\mathbf{C}$. A square is a set S of the form $S = \{z \in \mathbf{C} \mid |Re(z-p)| < \delta, |Im(z-p)| < \delta\}$. The point p is the center of S and δ its radius. Given a square S with center p and radius δ, then, given $k > 0$, denote by S^k the square with center p and radius $k\delta$.

Lemma 1. *Given $\varepsilon > 0$, $0 < k < 1$, $c > 0$ and $N > 0$, there exists $\delta = \delta(N, \varepsilon, k, c) > 0$ such that if S is s square of radius $< \delta$ such that $d(S,p) > c$ for every parabolic or attracting periodic point p, and, if for some $n > 0$, $V \in c(S,n)$ is such that $\Delta(V,n) \leq N$, then* $\operatorname{diam} W \leq \varepsilon$ *for all $W \in c(S^k, n)$ contained in V.*

Remark. Clearly, if follows from the lemma that $\operatorname{diam} f^i(W) \leq \varepsilon$ for all $0 \leq i \leq n$ because $f^i(V) \in c(S, n-i)$ and $f^i(W) \in c(S^k, n-i)$ and is contained in $f^i(V)$, which has at most N points x where $(f^{n-i})'(x) = 0$.

Proof. Suppose that for $\varepsilon > 0$, $0 < k < 1$, $c > 0$ and $N > 0$ the Lemma is false. Then, there exists a sequence S_n, $n = 1, 2, \ldots$ of squares with radius $\delta_n \to 0$ such that $d(S_n, p) > c$ for every parabolic or attracting periodic point p and sets $V_n \in c(S_n, m_n)$ satisfying $\Delta(V_n, m_n) \leq N$, but with $\operatorname{diam} W_n > \varepsilon$ for some $W_n \in c(S_n^k, m_n)$ contained in V_n. The condition $\delta_n \to 0$ implies that $m_n \to +\infty$. Take $0 < \gamma < 1$ such that $\gamma^{N+1} = k$. Then, there exists $0 \leq j \leq N+1$ such that, denoting by $\hat{S}_n = S_n^{\gamma^j}$, the set $\hat{S}_n - \hat{S}_n^\gamma$ doesn't contain critical values of f^{m_n}/V_n. Obviously, $\hat{S}_n^\gamma \supset S_n^k$. Take $\hat{W}_n \in c(\hat{S}_n^\gamma, m_n)$, such that $W_n \subset \hat{W}_n \subset V_n$. Then, $\operatorname{diam} \hat{W}_n > \varepsilon$.

Take $0 \leq j_n \leq m_n$ such that

$$\operatorname{diam} f^{j_n}(\hat{W}_n) > \varepsilon \tag{a}$$

$$\operatorname{diam} f^{j_n+1}(\hat{W}_n) \leq \varepsilon \tag{b}$$

observe that

$$f^{j_n}(\hat{W}_n) \in c(\hat{S}_n^\gamma, m_n - j_n).$$

Taking $\varepsilon > 0$ small enough, it follows from (a) and (b) that $f^{j_n}(\hat{W}_n)$ is a topological disk. Set $\tilde{W}_n = f^{j_n}(\hat{W}_n)$. Since $\hat{S}_n - \hat{S}_n^\gamma$ doesn't contain critical values of f^{m_n}/V_n, the element $\tilde{W}_n^+ \in c(\hat{S}_n, m_n - j_n)$ that contains

$\tilde{W}_n \in c(\hat{S}_n^k, m_n - j_n)$ is also a topological disk. Let $D = \{z \,|\, |z| < 1\}$ be the unit disk. Take conformal representations $\psi_n\colon D \to \hat{S}_n$, $\varphi_n\colon D \to \tilde{W}_n^+$ with $\psi_n(0) = x_n$, $f^{m_n - j_n}(\varphi_n(0)) = x_n$. Define:

$$F_n := \psi_n^{-1} f^{m_n - j_n} \varphi_n : D \hookleftarrow$$

Then $F_n : D \hookleftarrow$ is a Blaschke product of degree $\leq N$. Hence, $\{F_n\}$ is obviously a normal family and then there exists disks $D \supset D_1 \supset D_2$ centered at zero such that:

$$D_2 \supset F_n^{-1}(\psi_n^{-1} \hat{S}_n^k) \supset D_1.$$

Moreover,

$$\tilde{W}_n = \varphi_n F_n^{-1}(\psi_n^{-1} \hat{S}_n^k) \supset \varphi_n(D_1) \tag{1}$$

$$\varphi_n(D_2) \supset \varphi_n(F_n^{-1}(\psi_n^{-1} \hat{S}_n^k)) = \tilde{W}_n. \tag{2}$$

Now observe that $\{\varphi_n\}$ is a normal family because there exist three periodic orbits not intersecting S_n and then $\varphi_n(D)$ doesn't intersect them. Hence, we can assume that $\varphi_n \to \varphi$ uniformly on compact subsets of D and, by (2), φ is non constant because for all n:

$$\operatorname{diam} \varphi_n(D_2) \geq \operatorname{diam}(\tilde{W}_n) \geq \varepsilon.$$

From

$$0 = \lim_{n \to +\infty} \operatorname{diam}(\hat{S}_n) = \lim_{n \to +\infty} \operatorname{diam} f^{m_n - j_n}(\tilde{W}_n^+)$$

follows that

$$\tilde{W}_n^+ \subset J(f)^c \tag{4}$$

for all n. Moreover, (1) and the fact that φ is non constant imply that $\tilde{W}_{n_1} \cap \tilde{W}_{n_2} \neq \varnothing$ for some $n_1 < n_2$. Since $\tilde{W}_{n_1} = f^m(\tilde{W}_{n_2})$ for some m, this and (4) imply that there exists a connected component U of $J(f)^c$ such that

$$f^m(U) = U \tag{5}$$

and

$$U \supset \tilde{W}_n \qquad \forall n. \tag{6}$$

But (3) implies

$$\lim_{n \to +\infty} \operatorname{diam} f^{m_n - j_n}(\tilde{W}_n) = 0. \tag{7}$$

Moreover,

$$\inf_n \operatorname{diam}(\tilde{W}_n) > 0 \tag{8}$$

and

$$\lim_{n\to+\infty} d(x, f^{m_n - j_n}(\tilde{W}_n)) = 0. \tag{9}$$

From (5)–(9), it follows that x is either a sink or a parabolic point, thus contradicting the property $d(S_n, p) \geq c > 0$ for all n and every parabolic or attracting periodic point p.

Now let us prove Theorem II. If S is a square with radius δ, denote by $\mathcal{L}(S)$ the family of squares contained in $S^{3/2} - S$ and having radius $\delta/4$. Denote by $\mathcal{L}^*(S)$ the family of squares $S_0^{3/2}$ with $S_0 \in \mathcal{L}(S)$. Suppose that x is not a parabolic periodic point or is contained in the ω-limit set of a recurrent critical point. Then there exists $\delta_0 > 0$ such that

1) There is no critical point c of f such that there exist $0 < n_1 \leq n_2$ satisfying

$$d(f^{n_1}(c), c) < \delta_0$$
$$d(f^{n_2}(c), x) < \delta_0$$

2) $|x - p| > 10\delta_0$ for every parabolic or attracting periodic point p.

Given $\varepsilon > 0$ take $\varepsilon_1 > 0$ satisfying

3) $0 < \varepsilon_1 < \min\{\varepsilon/10, \delta_0/10\}$

4) If U is an open connected set with $\operatorname{diam} U \leq 2\varepsilon_1$ then $\operatorname{diam} W \leq \delta_0$ for all $W \in c(U, 1)$

Let N_0 be the number of critical points of f. Take $N_1 > 2$ such that

5) If S is a square and $V \in c(S, n)$ satisfies $\Delta(V, n) \leq N_0 + 1$ then the number of connected components of $f^{-n}(S^{2/3})$ contained in V is $\leq N_1$

Finally, take δ given by

6) $\delta = \min\{\delta_0/10, \varepsilon_1/10, \delta(2N_0, \frac{\varepsilon_1}{20N_1}, \frac{2}{3}, \delta_0)\}$ where $\delta(2N_0, \frac{\varepsilon_1}{20N_1}, \frac{2}{3}, \delta_0)$ is given by the lemma.

Let S_0 be the square of center x and radius δ. Suppose that Theorem II fails for $U = S_0$. Then there exist $n > 0$ and $V \in c(S_0, n)$ with $\operatorname{diam} V \geq \varepsilon \geq 10\varepsilon_1$. On the other hand, by (1), $\operatorname{diam} S_0 = 2\sqrt{2}\delta < 3\delta < \varepsilon_1$. Hence, there exists an integer $n_0 \geq 0$ such that there exists

$V_0 \in c(S_0^{3/2}, n_0)$ satisfying

7) $\operatorname{diam}(f^{-(n_0-i)}(S_0) \cap f^i(V_0)) \leq \varepsilon_1$ for all $1 \leq i \leq n_0$, and

8) $\operatorname{diam}(f^{-n_0}(S_0) \cap V_0) > \varepsilon_1$

Since $\operatorname{diam} S_0 < \varepsilon_1$ it follows that $n_0 > 0$. Now, starting with S_0 we shall construct a sequence of squares $S_0, S_1, S_2, \dots$ and strictly positive integers $n_0 \geq n_1 \geq n_2 \geq \dots$ satisfying

9) $S_{j+1} \in \mathcal{L}^*(S_j)$

10) There exists $V_j \in c(S_j^{3/2}, n_j)$ such that

$$\operatorname{diam}(f^{-(n_j-i)}(S_j) \cap f^i(V_j)) \leq \varepsilon_1$$

for all $1 \leq i \leq n_j$ and

$$\operatorname{diam}(f^{-n_j}(S_j) \cap V_j) > \varepsilon_1.$$

From (7) and (8), it follows that S_0 satisfies (10). If we construct such a sequence of squares and integers, then Theorem II will be proved by contradiction because the condition $n_0 \geq n_1 \geq \cdots \geq n_m \geq \cdots > 0$ implies that $n_j = n_i$ for all $j \geq i$ for a certain i. But (a) implies that the radius of S_j is $(3/8)^j$; in particular $\operatorname{diam} S_j \to 0$ when $j \to +\infty$. But by (10),

$$\varepsilon_1 < \operatorname{diam}(f^{-n_j}(S_j) \cap V_j) = \operatorname{diam}(f^{-n_i}(S_j) \cap V_j),$$
$$V_j \in c(S_j^{3/2}, n_j) = c(S_j^{3/2}, n_i).$$

Taking $j \to +\infty$, and recalling that i is constant and $\lim_{j\to+\infty} \operatorname{diam} S_j = 0$, we conclude that the inequality above cannot hold.

The sequences $\{S_j\}$ and $\{n_j\}$ will be constructed by induction starting with S_0. Suppose S_i and n_i constructed for $0 \leq i \leq j$. To find S_{j+1} and n_{j+1} we begin by observing that from (a) it follows that if $p \in S \in \mathcal{L}^*(S_j)$, then, by

$$\begin{aligned} d(p,x) &\leq \operatorname{diam} S + \sum_{i=0}^{j} \operatorname{diam} S_i \\ &= \sum_{i=0}^{j+1} (\frac{3}{8})^i \operatorname{diam} S_0 = 2\sqrt{2} \sum_{i=0}^{j+1} (\frac{3}{8})^i \delta \leq 4\sqrt{2}\delta. \end{aligned}$$

Hence, if a point q satisfies $d(q,S) \leq \delta_0$, we have

$$d(q,x) \leq 4\sqrt{2}\delta + \delta_0 \leq 2\delta_0.$$

By (2), this means that

11) $d(q,S) > \delta_0$ for all $S \in \mathcal{L}^*(S_j)$ and all parabolic or attracting periodic point q.

For the induction step (i.e. the construction of S_{j+1} and n_{j+1}), we shall use the following easy lemma.

Lemma 2. *If $U \subset \mathbf{C}$ is an open set and $V \in c(U,n)$ satisfies*

$$\operatorname{diam} f^i(V) \leq \delta_0 \qquad 0 \leq i \leq n$$

then

$$\Delta(V,n) \leq N_0.$$

Proof. If $\Delta(V,n) \geq N_0 + 1$, there exist $N_0 + 1$ different points x_i, $1 \leq i \leq N_0 + 1$, in V_j such that $(f^{n_j})'(x_j) = 0$. This means that for each $1 \leq i \leq N_0 + 1$ there exist $1 \leq m_i < n$, such that $f^{m_i}(x_i)$ is a critical point. Recalling that N_0 is the number of critical points of f, it follows that there exist two different points in the set $\{x_i | 1 \leq i \leq N_0 + 1\}$, that we shall denote by x_1, x_2, and a critical point c such that $f^{m_1}(x_1) = f^{m_2}(x_2) = c$. Assume $0 \leq m_1 \leq m_2$. Then

$$d(f^{m_2 - m_1}(c), c) = d(f^{m_2}(x_1), f^{m_2}(x_2)) \leq \operatorname{diam} f^{m_2}(V_j) \leq \delta_0$$

and

$$\begin{aligned} d(f^{n-m_1}(c), x) &= d(f^{n-m_1}(f^{m_1}(x_1)), x) \\ &= d(f^n(x_1), x) \leq \delta_0 \end{aligned}$$

contradicting property (1) of δ_0. □

Now, to find S_{j+1} and n_{j+1} we first claim that there exists $S \in \mathcal{L}(S_j)$ that for some $0 < n \leq n_j$ has $V \in c(S,n)$ with $\operatorname{diam} V \geq \varepsilon_1/10N_1$. Suppose that the claim is false. Then, for all $1 \leq i \leq n_j$,

$$\begin{aligned} \operatorname{diam} f^i(V_j) &\leq \operatorname{diam}(f^{-(n_j - i)}(S_j) \cap f^i(V_j)) \\ &\quad + \sup\{\operatorname{diam} W | W \in c(S, n_j - i), S \in \mathcal{L}(S_j)\} \\ &\leq \varepsilon_1 + \frac{\varepsilon_1}{10N_1} \leq 2\varepsilon_1. \end{aligned}$$

From this inequality applied to $i = 1$ and property (4), we have

$$\operatorname{diam} V_j \leq \delta_0.$$

Moreover, since $2\varepsilon_1 \leq \delta_0$ (by (3)),

$$\operatorname{diam} f^i(V_j) \leq \delta_0$$

for all $1 \leq i \leq n_j$, hence for all $0 \leq i \leq n_j$. By Lemma 2 this proves $\Delta(V_j, n_j) \leq N_0$.

Then, since $V_j \in c(S_j^{3/2}, n_j)$, it follows from (5), (11) and the lemma that

$$W \in c(S_j, n_j), W \subset V_j \Rightarrow \operatorname{diam} W \leq \varepsilon_1/10N_1.$$

Moreover, by the way N_1 was chosen, we have

$$\#\{W \in c(S_j, n_j) | W \subset V_j\} \leq N_1$$

and we are assuming that

$$S \in \mathcal{L}(S_j), U \in c(S, n_j) \Rightarrow \operatorname{diam} U \leq \varepsilon_1/10N_1.$$

Now observe that V_j is the union of sets $U \in c(S, n_j)$, $U \subset V_j$, $S \in \mathcal{L}(S_j)$ and the sets $W \in c(S_j, n_j)$, $W \subset V_j$. Moreover, for any two sets W', W'' in this family there exist $W' = W_0, W_1, \ldots, W_k = W''$ in $c(S_j, n_j)$ and contained in V_j such that for all $0 \leq i < k$ there exist $S_i \in \mathcal{L}(S_j)$ and $U_i \in c(S_i, n_j)$, such that $\overline{U}_i \cap \overline{W}_i \neq \phi$, $\overline{U}_i \cap \overline{W}_{i+1} \neq \phi$. Then

$$\operatorname{diam} V_j \leq N_1 \left(\frac{\varepsilon_1}{10N_1} + \frac{\varepsilon_1}{10N_1} \right) = \frac{\varepsilon_1}{5}$$

contradicting the last inequality in condition (11). This completes the proof of the claim. Now we can take $S \in \mathcal{L}(S_j)$ such that $\operatorname{diam} V \geq \varepsilon_1/10N_1$ for some $V \in c(S, n)$, $0 \leq n \leq n_j$. Take $\hat{V} \in c(S^{3/2}, n)$ containing V. Suppose that $\Delta(\hat{V}, n) \leq N_0$. Then, by Lemma 1 and condition (5)

$$\operatorname{diam} V \leq \varepsilon_1/20N_1$$

because $V \in c((S^{3/2})^{2/3}, n)$ and is contained in $\hat{V}$. This contradicts $\operatorname{diam} V \geq \varepsilon_1/10N_1$ and proves $\Delta(\hat{V}, n) \geq N_0 + 1$. From Lemma 2, it follows that

$$\operatorname{diam} f^i(\hat{V}) > \delta_0$$

for some $0 \leq i \leq n$. Now we define $S_{j+1} = S^{3/2}$. Then $f^i(\hat{V}) \in c(S^{3/2}, n-i)$ and $\operatorname{diam} f^i(\hat{V}) > \delta_0 \geq 10\varepsilon_1$. Moreover $\operatorname{diam} S^{3/2}_{j+1} \leq 2\delta < \varepsilon_1$. Then there exists $0 \leq n_{j+1} \leq n-i \leq n_j - i$ and $V_{j+1} \in c(S^{3/2}_{j+1}, n_{j+1})$ such that

$$\operatorname{diam}(f^{-n_{j+1}}(S_{j+1}) \cap V_{j+1}) > \varepsilon_1$$

and

$$\operatorname{diam}(f^{-n_{j+1}+i}(S_{j+1}) \cap f^i(V_{j+1})) \leq \varepsilon_1.$$

Observe that $n_{j+1} > 0$ because $\operatorname{diam} S_{j+1} < 2\delta < \varepsilon_1$. This completes the construction of the sequences $\{S_j\}$ and $\{n_j\}$ and the proof of part (a) of Theorem II.

Property (b) of Theorem II follows from (a) and Lemma 2. To prove (c), note that (b) implies that there exists $n_1 > 0$ such that $V \in c(U, n)$, $n \geq n_1$ implies that V doesn't intersect the forward orbits of the critical points. Now (c) follows from this fact and the following classical property (Fatou, loc. cit.): If V is an open set such that $V \cap J(f) \neq \varnothing$ and V doesn't intersect the forward orbit of the critical points, then given an open set $V_0 \subset \overline{V}_0 \subset V$ and $\varepsilon_0 > 0$ there exists $n_0 \geq 0$ such that every $W \in c(V_0, n)$, $n \geq n_0$, satisfies $\operatorname{diam}(W) \leq \varepsilon_0$. The proof of theorem is now complete.

Proof of the Corollary. To prove the corollary of Theorem II, suppose that Γ is the boundary of a Siegel disk or a connected component of a Herman ring, say, B, that, without loss of generality, we shall assume it satisfies $f(B) = B$, $f(\Gamma) = \Gamma$. Let $c_1, \ldots, c_k$ be the recurrent critical points of f. Set $\Gamma_i = \omega(c) \cap \Gamma$. Obviously

$$f(\Gamma_i) \subset \Gamma_i \qquad 1 \leq i \leq k. \tag{13}$$

Let us prove that

$$B \subset \bigcup_i \Gamma_i. \tag{14}$$

Suppose by contradiction that there exists

$$x \in B - \bigcup_i \Gamma_i.$$

Let U be the neighborhood of x given by Theorem II. Take $y \in B \cup U$ and take $\delta > 0$ such that $d(z, \Gamma) > \delta$ for all $y \in f^{-n}(x) \cap B$ and $n \geq 0$. By Theorem II there exists $n_0 \geq 0$ such that

$$V \in c(U, n), \quad n \geq n_0 \Rightarrow \operatorname{diam}(V) \leq \delta. \tag{15}$$

Take $z \in B$ such that $f^{n_0}(z) = y$. Let $V \in c(n_0, U)$ be such that $z \in V$. Since $\operatorname{diam}(V) \leq \delta$ and $d(z, \Gamma) > \delta$ it follows that $V \subset B$. Take $w \in V \subset B$ such that $f^{n_0}(w) = x \in \Gamma$. Then $w \in B$ and $w \in J(f)$. This contradiction proves (14). Now observe that Γ supports an invariant ergodic probability, namely that induced by the boundary map of the conformal representation $\varphi\colon A \to B$, where φ is an annulus or a disk (according to whether B is a Siegel disk or Herman ring), normalized, if B is a Siegel disk, by mapping the center of the disk in the fixed point contained in B. Then (13) and (14) imply $\Lambda_i = \Gamma$ for some $1 \leq i \leq k$, thus concluding the proof of the Corollary. □

Proof of Theorem I. To prove Theorem I consider a compact invariant set $\Lambda \subset J(f)$ not containing critical points, parabolic periodic points and not intersecting the ω-limit set of recurrent critical points. We want to prove that Λ is expanding. Suppose it is not. Then there exists a sequence of points $\{x_n\} \subset \Lambda$ such that $|(f^n)'(x_n)| \leq 1$ for all n. Take a subsequence $\{x_{n_j}\}$ such that the sequence $\{f^{n_j}(x_{n_j})\}$ converges to a point $x \in \Lambda$. Take $\varepsilon > 0$ such that $d(\Lambda, c) > 2\varepsilon$ for every critical point c. Let U be the neighborhood of x associated to $\varepsilon > 0$ given by Theorem II. Without loss of generality we can assume that U is a disk centered at x. For j large, we have $f^{n_j}(x_{n_j}) \in U$. Let $V_j \in c(U, n_j)$ be such that $x_{n_j} \in V_j$. Then $f^i(V_j) \in c(U, n_j - i)$ for all $0 \leq i \leq n_j$. By Theorem II, $\operatorname{diam} f^i(V_j) \leq \varepsilon$ for all $0 \leq i \leq n_j$. Hence $f^i(V_j)$ doesn't contain critical points for $0 \leq i \leq n_j$ thus implying that f^{n_j} has no critical points on V_j. Since U is a disk and $V_j \in c(U, n_j)$, it follows that $f^{n_j}\colon V_j \to U$ is a bijection. Let $\varphi_j\colon U \to V_j$ be its holomorphic inverse. Then the family $\{\varphi_j\}$ is normal because $\operatorname{diam} V_j \leq \varepsilon$ for all j. Therefore we can assume that it converges to a holomorphic map $\varphi: U \to \overline{\mathbf{C}}$. But

$$\left|\varphi_j'(f^{n_j}(x_{n_j}))\right| = \left|(f^{n_j})'(x_{n_j})\right|^{-1} \geq 1$$

and $\lim_{j \to +\infty} f^{n_j}(x_{n_j}) = x$. Then $|\varphi'(x)| \geq 1$ and φ is non constant. Moreover

$$\varphi(x) = \lim_{j \to +\infty} \varphi_j(f^{n_j}(x_{n_j})) = \lim_{j \to +\infty} x_{n_j} \in \Lambda.$$

Then $\varphi(U)$ covers a compact neighborhood W of $\varphi(x) \in \Lambda$. Since $x \in \Lambda \subset J(f)$ there exists $N > 0$ such that $f^n(W) \supset J(f)$ for all $n \geq N$. Since the univalent maps $\varphi_j: U \to \overline{\mathbf{C}}$ converge to $\varphi: U \to \overline{\mathbf{C}}$, it follows that $\varphi_j(U) \supset W$ for j large. Hence

$$f^{n_j}(W) \subset f^{n_j}\varphi_j(U) = U.$$

For j large, we have $n_j \geq N$ and then $f^{n_j}(W) \supset J(f)$. Hence $U \supset J(f)$ contradicting the fact that U can be taken arbitrarily small in diameter. The proof of Theorem I is complete. □

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